LAMINAR BOUNDARY LAYER ON A FLAT PLATE IN AN OSCILLATORY FLOW

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We consider the unsteady flow of an incompressible fluid in a laminary boundary layer formed in longitudinal flow over a flat plate where the external flow has a constant velocity on which a sinusoidal disturbance is imposed.

Two variants of the problem are considered. In one case the harmonic oscillations of the velocity of the incoming flow depend only on the time and in the other case they also depend on the longitudinal coordinate (disturbances conveyed by the flow).

The assumption of relative smallness of the amplitude of the disturbances means that the solution of the problem can be derived in the form of a power series and the coefficients of the first three terms of these series are sought. The problem is reduced to the determination of universal functions by a special choice of dimensionless variables.

The stability of the flow in the boundary layer in the second case is investigated by means of the energy method of the theory of hydrodynamic stability.

1. Velocity Distribution in Boundary Layer in the Case of Periodic Disturbances in the External Flow. We consider two variants of this problem. We put the velocity of the incoming flow in one case (case I) in the form

$$U(x_1, t) = U_0 [1 + \lambda \cos \omega (x_1 / U_0 - t)] \tag{1.1}$$

and in the second case (case II) in the form

$$U(t) = U_0 (1 + \lambda \cos \omega t) \tag{1.2}$$

(the x_1 axis is directed along the plate; t is the time). Expression (1.1) corresponds to an external flow with constant velocity U_0 on which are imposed sinusoidal disturbances with amplitude λ and frequency ω, which are carried away by the external flow with inflow velocity U_0. This problem was investigated on the assumption that λ was small in [1], where a solution for small frequencies ω was given.

In the second case the imposed sinusoidal disturbances in the external flow do not depend on the spatial coordinate x_1. In [2] a review of the theoretical investigations and the results of thorough experimental investigation of such flow in a boundary layer were given. It should be noted, however, that the theoretical investigation of flows has been confined to the region of either small or large frequencies of disturbances.

In this paper we impose no restriction on the frequency ω, but we assume that the amplitude $\lambda \ll 1$.

The equations of an unsteady two-dimensional boundary layer have the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} + v\frac{\partial u}{\partial x_2} = -\frac{1}{\rho}\frac{\partial p}{\partial x_1} + \nu\frac{\partial^2 u}{\partial x_2^2}, \quad \frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial x_2} = 0, \tag{1.3}$$

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where

$$-\frac{1}{\rho}\frac{\partial p}{\partial x_1}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x_1}$$

The boundary conditions are

$$\begin{aligned}
&u(x_1, x_2, t) = v(x_1, x_2, t) = 0 \quad (x_2 = 0)\\
&u(x_1, x_2, t) \to U(x_1, t) \qquad (x_2 \to \infty) \qquad\qquad (\mathrm{I})\\
&u(x_1, x_2, t) \to U(t) \qquad (x_2 \to \infty) \qquad\qquad (\mathrm{II})
\end{aligned}$$

Here $u(x_1, x_2, t)$ and $v(x_1, x_2, t)$ are the longitudinal and transverse components, respectively, of the velocity in the boundary layer, p is the pressure, ν is the kinematic viscosity, and the x_2 axis is perpendicular to the plate.

Assuming $\lambda \ll 1$, we will seek the solution of Eqs. (1.3) in the form

$$\begin{aligned}
u(x_1, x_2, t) &= u_0(x_1, x_2) + \lambda u_1(x_1, x_2, t) + \lambda^2 u_2(x_1, x_2, t) + \ldots\\
v(x_1, x_2, t) &= v_0(x_1, x_2) + \lambda v_1(x_1, x_2, t) + \lambda^2 v_2(x_1, x_2, t) + \ldots
\end{aligned} \qquad (1.4)$$

In what follows we will determine only the first three terms of these series.

We consider case I, where the velocity of the external flow is given by expression (1.1). Substituting expressions (1.4) in Eqs. (1.3) and equating the coefficients of equal powers of λ to zero we obtain a system of equations for the coefficients of the expansions (1.4).

The terms with zero power of λ give the steady-state equations

$$u_0\frac{\partial u_0}{\partial x_1}+v_0\frac{\partial u_0}{\partial x_2}=\nu\frac{\partial^2 u_0}{\partial x_2^2}, \quad \frac{\partial u_0}{\partial x_1}+\frac{\partial v_0}{\partial x_2}=0 \qquad (1.5)$$

with boundary conditions

$$u_0 = v_0 = 0 \quad (x_2 = 0), \qquad u_0 \to U_0 \quad (x_2 \to \infty)$$

Terms with λ^1 give the equations

$$\begin{aligned}
&\frac{\partial u_1}{\partial t}+u_0\frac{\partial u_1}{\partial x_1}+v_0\frac{\partial u_1}{\partial x_2}+u_1\frac{\partial u_0}{\partial x_1}+v_1\frac{\partial u_0}{\partial x_2}=\nu\frac{\partial^2 u_1}{\partial x_2^2}\\
&\frac{\partial u_1}{\partial x_1}+\frac{\partial v_1}{\partial x_2}=0
\end{aligned} \qquad (1.6)$$

with boundary conditions

$$u_1 = v_1 = 0 \quad (x_2 = 0), \qquad u_1 \to U_0\cos\omega\left(\frac{x_1}{U_0}-t\right) \quad (x_2 \to \infty)$$

Terms with λ^2 give the equations

$$\begin{aligned}
&\frac{\partial u_2}{\partial t}+u_0\frac{\partial u_2}{\partial x_1}+v_0\frac{\partial u_2}{\partial x_2}+u_2\frac{\partial u_0}{\partial x_1}+v_2\frac{\partial u_0}{\partial x_2}+u_1\frac{\partial u_1}{\partial x_1}+v_1\frac{\partial u_1}{\partial x_2}=\\
&\qquad=\nu\frac{\partial^2 u_2}{\partial x_2^2}-\frac{U_0\omega}{2}\sin 2\omega\left(\frac{x_1}{U_0}-t\right)\\
&\qquad\frac{\partial u_2}{\partial x_1}+\frac{\partial v_2}{\partial x_2}=0
\end{aligned} \qquad (1.7)$$

with boundary conditions

$$u_2 = v_2 = 0 \quad (x_2 = 0), \qquad u_2 \to 0 \quad (x_2 \to \infty)$$

We note that the first terms in the expansions (1.4), as was to be expected, are a known solution of the Blasius problem of the boundary layer on a flat plate:

$$u_0 = U_0 f'(\eta), \quad v_0 = \frac{1}{2}\left(\frac{\nu U_0}{x_1}\right)^{1/2} [\eta f'(\eta) - f(\eta)] \qquad \left(\eta = x_2\left(\frac{U_0}{\nu x_1}\right)^{1/2}\right) \tag{1.8}$$

Tables for the function $f(\eta)$ and its derivative $f'(\eta)$ are given in [3].

In Eqs. (1.6) and (1.7) we convert to dimensionless variables

$$\eta = x_2\left(\frac{U_0}{\nu x_1}\right)^{1/2}, \qquad \zeta = \frac{\omega x_1}{U_0}, \qquad \psi = \omega\left(\frac{x_1}{U_0} - t\right)$$
$$u_i = U_0 u_i^\circ, \qquad v_i = \left(\frac{\nu U_0}{x_1}\right)^{1/2} v_i^\circ \quad (i = 1,2) \tag{1.9}$$

Equations (1.6) in the new variables have the form

$$\frac{\partial^2 u_1^\circ}{\partial \eta^2} + \zeta(1-f')\frac{\partial u_1^\circ}{\partial \psi} - \zeta f' \frac{\partial u_1^\circ}{\partial \zeta} + \frac{f}{2}\frac{\partial u_1^\circ}{\partial \eta} + \frac{1}{2} f''\eta u_1^\circ - f'' v_1^\circ = 0$$
$$\zeta\left(\frac{\partial u_1^\circ}{\partial \psi} + \frac{\partial u_1^\circ}{\partial \zeta}\right) - \frac{\eta}{2}\frac{\partial u_1^\circ}{\partial \eta} + \frac{\partial v_1^\circ}{\partial \eta} = 0 \tag{1.10}$$

with boundary conditions

$$u_1^0 = v_1^0 = 0 \quad (\eta = 0), \qquad u_1^0 \to \cos\psi \quad (\eta \to \infty)$$

Equations (1.7) in these variables are

$$\frac{\partial^2 u_2^\circ}{\partial \eta^2} + \zeta(1-f')\frac{\partial u_2^\circ}{\partial \psi} - \zeta f' \frac{\partial u_2^\circ}{\partial \zeta} + \frac{1}{2} f \frac{\partial u_2^\circ}{\partial \eta} + \frac{1}{2} f''\eta u_2^\circ - f'' v_2^\circ =$$
$$= \frac{\eta}{2} u_1^\circ \frac{\partial u_1^\circ}{\partial \eta} - \zeta u_1^\circ\left(\frac{\partial u_1^\circ}{\partial \psi} + \frac{\partial u_1^\circ}{\partial \zeta}\right) - v_1^\circ \frac{\partial u_1^\circ}{\partial \eta} - \frac{\zeta}{2}\sin 2\psi$$
$$\zeta\left(\frac{\partial u_2^\circ}{\partial \psi} + \frac{\partial u_2^\circ}{\partial \zeta}\right) - \frac{\eta}{2}\frac{\partial u_2^\circ}{\partial \eta} + \frac{\partial v_2^\circ}{\partial \eta} = 0 \tag{1.11}$$

with boundary conditions

$$u_2^0 = v_2^0 = 0 \quad (\eta = 0), \qquad u_2^0 \to 0 \quad (\eta \to \infty)$$

Using the method of separation of the variables in view of the fact that Eqs. (1.10) are linear and homogeneous and the coefficients are independent of the variable Ψ, we will seek the solutions of Eqs. (1.10) in the form

$$u_1^\circ = A_{10}(\eta, \zeta)\cos\psi + B_{10}(\eta, \zeta)\sin\psi$$
$$v_1^\circ = C_{10}(\eta, \zeta)\cos\psi + D_{10}(\eta, \zeta)\sin\psi \tag{1.12}$$

Expressions (1.12) can be written in another form

$$u_1^0 = a\cos(\psi + \alpha), \qquad v_1^0 = b\cos(\psi + \beta)$$
$$a = \sqrt{A_{10}^2 + B_{10}^2}, \quad b = \sqrt{C_{10}^2 + D_{10}^2}, \quad \operatorname{tg}\alpha = -B_{10}/A_{10}, \quad \operatorname{tg}\beta = -D_{10}/C_{10}$$

We substitute (1.12) in (1.10), group the terms, and equate the coefficients of $\cos\Psi$ and $\sin\Psi$ to zero. We obtain a system of four partial differential equations of the mixed type

$$L(A_{10}) = -\zeta(1-f')B_{10} + f''C_{10}, \quad L(B_{10}) = \zeta(1-f')A_{10} + f''D_{10} \quad K(C_{10}, A_{10}) = -\zeta B_{10}, \quad K(D_{10}, B_{10}) = \zeta A_{10} \tag{1.13}$$

with boundary conditions

$$A_{10} = B_{10} = C_{10} = D_{10} = 0 \quad (\eta = 0), \quad A_{10} \to 1, \quad B_{10} \to 0 \quad (\eta \to \infty)$$

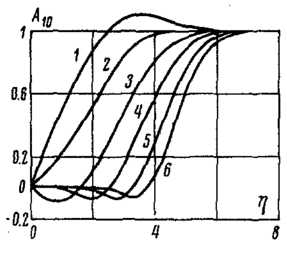

Fig. 1

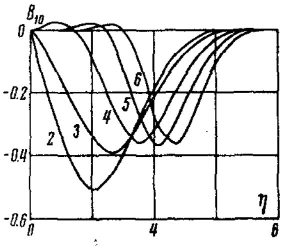

Fig. 2

Here

$$L(p) = \frac{\partial^2 p}{\partial \eta^2} - \zeta f' \frac{\partial p}{\partial \zeta} + \frac{f}{2}\frac{\partial p}{\partial \eta} + \frac{\eta}{2} f'' p$$
$$K(p, q) = \frac{\partial p}{\partial \eta} - \frac{\eta}{2}\frac{\partial q}{\partial \eta} + \zeta \frac{\partial q}{\partial \zeta}$$

We seek the solution of Eqs. (1.11) in the form

$$\begin{aligned} u_2^\circ &= A_{20}(\eta, \zeta) + A_{21}(\eta, \zeta)\cos 2\psi + B_{21}(\eta, \zeta)\sin 2\psi \\ v_2^\circ &= C_{20}(\eta, \zeta) + C_{21}(\eta, \zeta)\cos 2\psi + D_{21}(\eta, \zeta)\sin 2\psi \end{aligned} \tag{1.14}$$

We substitute (1.12) and (1.14) in (1.11) and equate the coefficients of cos 2Ψ and sin 2Ψ and the sum of the remaining terms to zero. As a result we obtain

$$\begin{aligned} L(A_{21}) &= -2\zeta(1 - f')B_{21} + f''C_{21} + \zeta A_{10}B_{10} + \frac{\zeta}{4}\frac{\partial}{\partial \zeta}(A_{10}^2 - B_{10}^2) - \\ &\quad - \frac{\eta}{8}\frac{\partial}{\partial \eta}(A_{10}^2 - B_{10}^2) + \frac{1}{2}\left(C_{10}\frac{\partial A_{10}}{\partial \eta} - D_{10}\frac{\partial B_{10}}{\partial \eta}\right) \\ L(B_{21}) &= 2\zeta(1 - f')A_{21} + f''D_{21} + \frac{\zeta}{2}(B_{10}^2 - A_{10}^2) + \\ &\quad + \frac{\zeta}{2}\frac{\partial}{\partial \zeta}(A_{10}B_{10}) - \frac{\eta}{4}\frac{\partial}{\partial \eta}(A_{10}B_{10}) + \frac{1}{2}\left(D_{10}\frac{\partial A_{10}}{\partial \eta} + C_{10}\frac{\partial B_{10}}{\partial \eta}\right) + \frac{\zeta}{2} \\ L(A_{20}) &= f''C_{20} - \frac{\eta}{8}\frac{\partial}{\partial \eta}(A_{10}^2 + B_{10}^2) + \frac{\zeta}{4}\frac{\partial}{\partial \zeta}(A_{10}^2 + B_{10}^2) + \frac{1}{2}\left(C_{10}\frac{\partial A_{10}}{\partial \eta} + D_{10}\frac{\partial B_{10}}{\partial \eta}\right) \\ K(C_{21}, A_{21}) &= -2\zeta B_{21}, \quad K(D_{21}, B_{21}) = 2\zeta A_{21}, \quad K(C_{20}, A_{20}) = 0 \end{aligned} \tag{1.15}$$

with boundary conditions

$$\begin{aligned} &A_{20} = A_{21} = B_{21} = C_{20} = C_{21} = D_{21} = 0 \qquad (\eta = 0) \\ &A_{20} \to 0, \quad A_{21} \to 0, \; B_{21} \to 0 \qquad (\eta \to \infty) \end{aligned}$$

Systems (1.13)-(1.15) were solved by a numerical method by using an implicit finite-difference scheme.

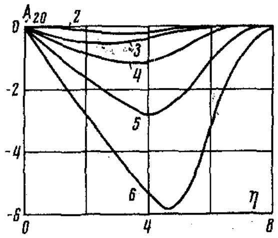

Fig. 3

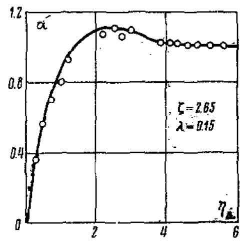

Fig. 4

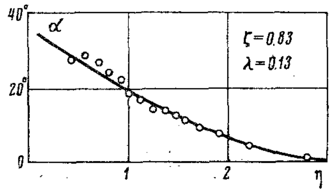

Fig. 5

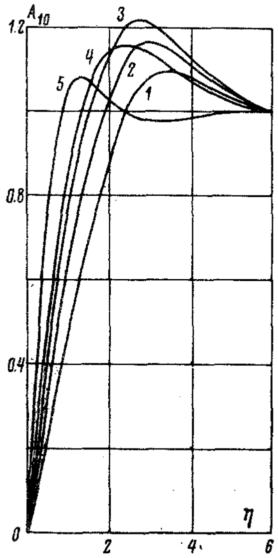

Fig. 6

The partial derivatives were approximated in the following way:

$$\frac{\partial q}{\partial \eta} \approx \frac{1}{\Delta\eta}\left[\theta\left(q_{i+1}{}^{n+1} - q_i{}^{n+1}\right) + (1-\theta)\left(q_{i+1}{}^{n} - q_i{}^{n}\right)\right] \qquad (1/2 \leqslant \theta \leqslant 1)$$

$$\frac{\partial q}{\partial \zeta} \approx \frac{1}{\zeta_{n+1} - \zeta_n}\left[\frac{1}{2}\left(q_{i+1}{}^{n+1} + q_i{}^{n+1}\right) - \frac{1}{2}\left(q_{i+1}{}^{n} + q_i{}^{n}\right)\right] \tag{1.16}$$

The boundary conditions for $\eta \to \infty$ were transferred to the line $\eta=10$. The line $\zeta=0$ will be the characteristic for the system of equations (1.13), (1.15). The calculation was made for each layer ζ=const by using the matrix elimination method, and in this problem we managed to reduce the order of the matrix to two. The value of θ was selected from a consideration of the solutions of the simplified model problem with different θ. All the calculations of the main problem were made with $\theta=1$. The problem was solved on an M-20 computer.

The above-described method and the sequence of operations in the solution of the problem can also be applied to case II, but on conversion to dimensionless variables in this case φ must be replaced by $\varphi=\omega t$. The nature of the equations of expansion (1.4) are also sought in the form (1.12) and (1.14).

By analyzing the behavior of the obtained universal functions in each case we can draw some qualitative conclusions regarding the properties of the considered flows. We will discuss case I first.

The graphs of the functions A_{10}, B_{10}, and A_{20} for this case are shown in Figs. 1-3, where the curves correspond to the following values of ζ:

Curves	1	2	3	4	5	6
	$\zeta=0$	0.87	2.20	10.01	30.53	72.70

From an examination of these curves and from the behavior of other functions not shown here, we can draw the following conclusions for case I.

1. With increase in the parameter $\zeta=\omega x_1/u_0$ the zone of the effect of external disturbances moves closer and closer to the outer boundary of the boundary layer.

2. A comparison of the maximum absolute values of functions A_{10}, B_{10}, A_{20}, A_{21}, and B_{21} for different ζ shows that with increase in ζ these values for A_{20}, A_{21}, and B_{21} increase rapidly, where for A_{10} and B_{10} they are almost constant. This means, in particular, that for a fixed frequency ω and velocity U_0 the contribution of the second frequency 2ω increases with increase in x_1.

3. The longitudinal velocity $\bar{u}(x_1, x_2)$, averaged over the period $2\pi/\omega$, of the boundary layer is given by

$$\frac{\bar{u}}{U_0} = f'(\eta) + \lambda^2 A_{20}(\eta, \zeta) \tag{1.17}$$

Function A_{20} (η, ζ), beginning at certain values of ζ (at approximately $\zeta=0.8$), is always nonpositive. Numerical analysis shows that for sufficiently large values of ζ points of inflection appear on the profiles of the averaged velocity.

The behavior of the boundary layer in case II was considered in detail in [2]. Examples of comparison of the obtained results with the experimental data [2] are given in Figs. 4 and 5. The relationships given represent, respectively, the amplitude for $\zeta=2.65$, $\lambda=0.15$ and the phase for $\zeta=0.83$, $\lambda=0.13$ of velocity $u_1{}^0$.

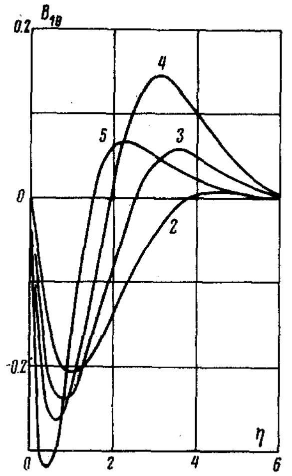

Fig. 7

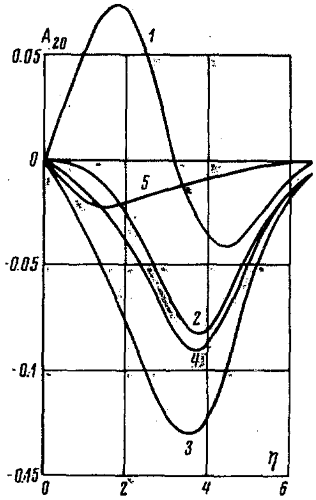

Fig. 8

For case II the functions A_{10}, B_{10}, and A_{20} are represented in Figs. 6-8, where the curves correspond to the following values of ζ:

Curves	1	2	3	4	5
ζ	=0	0.67	1.21	2.01	6.12

It should be noted that the behavior of the flow in the boundary layer in the two considered cases is significantly different.

1. In case II, as distinct from case I, the zone of effect of external disturbances moves closer and closer to the plate with increase in ζ.

2. In case I the maximum amplitude a of the oscillations is attained when $\zeta=0$, whereas in case II it is attained when $\zeta \approx 1.5$.

3. At sufficiently large values of ζ in case II the effect of oscillations in the external flow on the profile of the averaged velocity in the boundary layer is reduced. This can be seen from the fact that the maximum absolute value of the function A_{20}, beginning at approximately $\zeta=1.5$, decreases. In this case inflection points cannot appear on the profiles of the averaged velocity.

2. Investigation of Flow Stability in Boundary Layer (for Case I). The main flow for which we investigate the stability is the flow, obtained in Par. 1, in the boundary layer of a flat plate when the external flow is given by expression (1.1). We will base our application of the energy method on [4], in which the stability of unsteady parallel flows was considered. As is usually done in the consideration of flow stability in a boundary layer, we neglect the inhomogeneity of the flow along its length and the transverse velocity component and regard the flow in the boundary layer as approximately parallel. The integral equation for the energy of the disturbed motion in some limited volume V of the flow field (using tensor notation) can be written as

$$\frac{d}{dt}\iiint\limits_V \frac{w_i w_i}{2}\,dV = -\iiint\limits_V w_i w_j \frac{\partial U_i}{\partial x_j}\,dV - \nu \iiint\limits_V \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\,dV$$
$$dV = dx_1 dx_2 dx_3 \qquad (2.1)$$

Here we sum over all the recurrent subscripts; ω_i is the velocity of the disturbance, and U_i is the velocity of the main flow.

This equation is obtained directly from the Navier-Stokes equations without any assumptions regarding the magnitude of the disturbances.

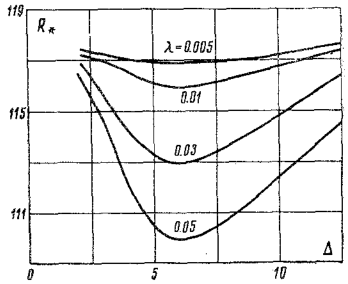

Fig. 9

We use a very simple "quasi-steady" determination of the stability of the unsteady flow. The unsteady flow at an indicated instant is regarded as stable if the kinetic energy of the disturbances of the instantaneous velocity field of the main flow decreases with time.

Only the first integral on the right side of Eq. (2.1) can be positive and, hence, according to this definition the flow is stable if

$$\nu \iiint\limits_V \frac{\partial w_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} dV \geqslant - \iiint\limits_V w_i w_j \frac{\partial U_i}{\partial x_j} dV \tag{2.2}$$

The case where the equality occurs is called the neutral stability case. It is of most interest.

With the assumptions stated above the main flow is given in the form

$$U_1 = U_3 = 0, \qquad U_2 = U_2(x_2, t)$$

We convert to dimensionless variables

$$w_i^\circ = \frac{w_i}{U_0}, \quad u^\circ = \frac{U_2}{U_0}, \quad y_i = \frac{x_i}{\delta}, \quad R = \frac{U_0 \delta}{\nu} \tag{2.3}$$

Here δ is the thickness of the boundary layer, defined as the distance from the wall to the point where the velocity $U_2=0.999$ U. Then condition (2.2) for the neutral stability case can be put in the form

$$R = -\left(\iiint\limits_V \frac{\partial w_i^\circ}{\partial y_j} \frac{\partial w_i^\circ}{\partial y_j} dV\right) \left(\iiint\limits_V w_1^\circ w_2^\circ \frac{\partial U}{\partial y_2} dV\right)^{-1} \tag{2.4}$$

If we are seeking only sufficient stability conditions the lower limit of the critical Reynolds number R_* can be determined by minimizing R in Eq. (2.4).

As in [4], we confine ourselves to a consideration of two-dimensional disturbances of the form

$$w_1^\circ = s(y_2, t)\, e^{i\gamma y_1}, \qquad w_2^\circ = z(y_2, t)\, e^{i\gamma y_1} \tag{2.5}$$

As a result we obtain a linear differential equation for z with homogeneous boundary conditions

$$z'''' - 2\gamma^2 z'' + \gamma^4 z = -i\gamma R \left(\frac{\partial u^\circ}{\partial y_2} z' + \frac{1}{2} \frac{\partial^2 u^\circ}{\partial y_2^2} z\right)$$
$$z = z' = 0 \quad (y_2 = 0), \qquad z \to 0, \quad z' \to 0 \quad (y_2 \to \infty) \tag{2.6}$$

where the time t plays the role of a parameter. The coefficients depend on the independent variable y_2 and the parameters γ and t.

Thus, we obtain a problem for the eigenvalues and we define the numbers $R=R(\gamma,\ t)$ as eigenvalues.

In this work the eigenvalues were determined numerically by the finite-difference method. Since R is minimized, only the smallest eigenvalues are of interest. The required value of R_* for each instant will be the smallest of the numbers R over the whole spectrum of γ.

Examining these smallest values of R_* for different instants (within one period in the considered problem) we find the lower limit of R_*.

We turn our attention to a shortcoming, noted in [4], of the method used. It is impossible to guarantee that the flow characteristics, particularly R, found in this way will satisfy the Navier-Stokes equations continuously in time. Strictly speaking, the velocities satisfy only the integral equation of the disturbance energy and the continuity equation.

An analysis of the effect of the frequency ω on the critical Reynolds number R_* showed that the smallest value of this number occurs at the same (irrespective of the amplitude of the oscillations) value of the dimensionless parameter $\Delta = \delta\sqrt{\omega/2\nu}$ (Fig. 9).

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